APPLIED MATHEMATICS and STATISTICS

DOCTORAL QUALIFYING EXAMINATION

in COMPUTATIONAL APPLIED MATHEMATICS

Summer 2025 (May)

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME: ______________________________

STUDENT ID: ______________________________

SIGNATURE: ______________________________

This is a closed-book exam. No calculator is allowed. Start your answer on its corresponding question page. If you use extra pages, print your name and the question number clearly at the top of each extra page. Hand in all answer pages.

Date of Exam: May 28th, 2025

Time: 9:00 AM – 1:00 PM

A1. Consider a differential operator L. The adjoint operator is defined as the operator L' such that

$$\int_\Omega v(x)\, L[u(x)]\, d\Omega = \int_\Omega L'[v(x)]\, u(x)\, d\Omega$$

for all v $\in C_0^m(\Omega)$, the set of m-times continuously differentiable functions with compact support in Ω.

(a) Find the adjoint operator for the following:

1) $\frac{\partial}{\partial t} + a(x)\,\frac{\partial}{\partial x}$

2) $\frac{\partial^2}{\partial t^2} - c^2\,\frac{\partial^2}{\partial x^2}$

3) $\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}$

4) $\frac{\partial}{\partial t} - \nu(t)\frac{\partial^2}{\partial x^2}$

(b) The weak (or distributional) derivative of a function f is a distribution f' defined by the equation $\langle f', v\rangle = -\langle v', f\rangle$. Find the weak derivative of

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } 0 \le x < 1, \\ 1 & \text{if } 1 \le x < 2, \\ 2 & \text{if } x \ge 2. \end{cases}$$

Continue solution:

A2. For the conservation law $u_t + f(u)_x = 0$, given the initial condition as

$$u(x,0) = \begin{cases} u_l & x < -1 \\ u_m & -1 < x < 1 \\ u_r & x > 1 \end{cases}$$

where $f(u) = u(10-u)$ and $u > 0$:

(a) Draw the characteristics and find solution $u(x,t)$ if $u_l = u_m = 2$ and $u_r = 8$.

(b) Draw the characteristics and find solution $u(x,t)$ if $u_l = 8$ and $u_m = u_r = 2$.

(c) Draw the characteristics and find solution $u(x,t)$ if $u_l = 2$ and $u_m = 5$ and $u_r = 8$.

Continue solution:

A3. Solve Laplace equation with the given boundary conditions. Find the maximum and minimum of the solution.

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in \Omega : x \in (0, 1), y \in (0, 1).$$

$$\begin{aligned} u(x, 0) &= 0, \\ u(x, 1) &= 4x(1 - x), \\ u(0, y) &= \sin 2\pi y, \\ u(1, y) &= \sin \pi y. \end{aligned}$$

Continue solution:

A4. Let f be analytic on a region A except for poles at $b_1, \ldots, b_m$, counted with their multiplicities, and let $a_1, \ldots, a_n$ be zeroes of f counted with their multiplicities (that is, if b_l is a pole of order k, then b_l is to be repeated k times in the list, and similarly for the zeros a_j). Let γ be a closed curve homotopic to a point in A and passing through none of the points a_j or b_l. Prove that

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 2\pi i \left[\sum_{j=1}^{n} I\left(\gamma; a_j\right) - \sum_{l=1}^{m} I\left(\gamma; b_l\right) \right].$$

Continue solution:

A5. Find an explicit conformal map from region A_1 to region A_2, where

$$A_1 = \{z \in \mathbb{C},\, \mathrm{Re}(z) > 0,\, \mathrm{Im}(z) > 0,\, |z| > 1\},$$

$$A_2 = \{z \in \mathbb{C},\, \mathrm{Im}(z) > 0\}.$$

Continue solution:

B6. Newton's method for solving nonlinear equations (10 points)

(a) (3 points) Derive Newton's iteration formula $x_{k+1} = x_k - f(x_k)/f'(x_k)$ using a first-order Taylor expansion of $f(x)$ around x_k. Give a geometric interpretation and state the generalization for systems $\boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}$, specifying the linear system solved at each iteration.

(b) (4 points) For a simple root x_* of $f(x) = 0$ ($f(x_*) = 0$, $f'(x_*) \neq 0$), prove that Newton's method exhibits Q-quadratic convergence near x_*. You may use the Taylor expansion $f(x_*) = f(x_k) + f'(x_k)(x_* - x_k) + \frac{1}{2}f''(\xi_k)(x_* - x_k)^2$. How does convergence change for multiple roots? State the Q-order of convergence and briefly justify your answer.

(c) (3 points) Provide an example where Newton's method fails to converge. Explain how a line search strategy with a merit function (e.g., $\phi(x) = \frac{1}{2}f(x)^2$) enhances robustness and global convergence.

Continue solution:

B7. Numerical solutions of initial value problems (10 points)

(a) (3 points) For the IVP $y'(t) = f(t, y(t))$, $y(t_0) = y_0$: Derive the trapezoidal rule by approximating $y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(t, y(t))dt$ with trapezoidal quadrature. State the order of accuracy (in terms of the global order) for both the trapezoidal rule and forward Euler method.

(b) (4 points) For the test equation $y'(t) = \lambda y(t)$ ($\lambda \in \mathbb{C}$) with $z = h\lambda$, find the stability function $R(z)$ for forward Euler and trapezoidal methods. Define A-stability and determine which of these two methods is A-stable.

(c) (3 points) Characterize stiff ODE systems. Between the two methods, which one is more suitable for stiff problems? Explain why. Discuss the implications of stiffness on the choice of time step size and the potential for numerical instability in explicit methods.

Continue solution:

B8. Consider a one-dimensional diffusion equation $v_t = \nu v_{xx}$, $\nu > 0$, on infinite interval and derive the leapfrog scheme. It is known that the leapfrog scheme is unconditionally unstable for this PDE.

(a) Replace u_k^n in the leapfrog scheme with the time average $(u_k^{n+1} + u_k^{n-1})/2$ to derive the so-called Dufort-Frankel scheme. Here u_k^n is a numerical approximation to the analytic solution $v(x,t)$ at point $(k\Delta x, n\Delta t)$.

(b) Investigate stability of the Dufort-Frankel scheme by using the discrete Fourier transform.

Continue solution:

B9. Consider a nonlinear scalar hyperbolic PDE in the conservation law form

$$v_t + [f(v)]_x = 0,$$

and write down the general expression for its conservative numerical scheme (you may assume that a 3-point stencil is used).

(a) Assuming that the numerical flux $F(.,.)$ is differentiable with respect to each argument, show that consistency requires the numerical flux F to be identical to the analytical flux function f for constant flows: if $v(x,t) = \bar{v}$, then $F(\bar{v}, \bar{v}) = f(\bar{v})$.

(b) Derive the Lax-Wendroff scheme for $v_t + [f(v)]_x = 0$ and find the numerical flux. Does it satisfy the consistency condition of (a)?

Continue solution:

B10. Using the method of modified PDE's investigate numerical dissipation and dispersion properties of the leapfrog scheme for the linear advection equation $v_t + av_x = 0$.

Continue solution: