Quantitative Finance Qualifying Exam

May 2025

Instructions: (1) You have 4 hours to do this exam. (2) This exam is closed notes and closed books. No electronic devices are permitted. (3) Phones must be turned completely off during the exam. (4) All problems are weighted equally.

Part 1: Do 2 out of problems 1, 2, 3. (AMS511)
Part 2: Do 2 out of problems 4, 5, 6. (AMS512)
Part 3: Do 2 out of problems 7, 8, 9. (AMS513)
Part 4: Do 2 out of problems 10, 11, 12. (AMS517)

Problems to be graded: Please write down which eight problems you want graded here.

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Name (Print clearly):

Student ID:

Signature:

Stony Brook University
Applied Mathematics and Statistics

1.

You are given the following information. Assume annual compounding throughout.

- A 1-year zero-coupon bond with a face value of $10,000 sells at a discount of $9,950.
- A 2-year bond with a face value of $10,000 and an annual coupon of $350 sells at premium of $10,500.
- A 3-year bond with a face value of $100,000 and an annual coupon of $4,000 sells at a premium of $105,000.

Solve for the following:

- Using the market data above, bootstrap the 3-year spot curve.
- Using the spot rates computed above compute the price of a 3-year bond with a face value of $100,000 and annual coupon of $450.
- Compute the forward rate $f_{2,3}$.

2.

An asset's price S(t) follows the dynamics of the constant coefficient geometric process described by the stochastic differential equation:

$$d S(t) = \mu S(t)\, dt + \sigma S(t)\, d W(t)$$

Carrying costs are continuously charged at the rate c and dividends continuously accrue at the rate b, both proportionally to the current price. Denote the risk-free rate of return by r_f.

Derive the SDE which describes the price dynamics under the risk-neutral measure. Fully explain your logic in deriving the solution.

3.

An investment universe of equities has returns which follow a multivariate Normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Let r_f denote the risk-free rate of return and $\mathbf{x}$ the allocation vector. Assume that the notional total capital is 1 and that short positions are permitted,

Derive a closed-form expression that computes the mean-variance (Markowitz) portfolio representing the tangent portfolio. Show all work supporting your answer.

4.

A distribution is said to have a power law tail if its survival function has the form:

$$\text{Prob}[R > r] = 1 - F(r) = L(r)\, r^{-\alpha}, \;\; \alpha > 0$$

where $F(r)$ is the CDF of R and $L(r)$ is a function of r meeting certain conditions.

- What conditions must hold for the function $L(r)$ in order for R to have a power law tail.
- For a return distribution with a power law tail, demonstrate mathematically which moments of R exist depending upon the value of the tail exponent.
- Given a sample of data, describe how an appropriately constructed plot of the survival function can be used to identify if a power law tail appears to exist and, if so, how that plot can be used to determine which values of R follow a power law and estimate the value of α.

5.

Consider a set of n assets $i \in \{1, 2, \ldots, n-1, n\}$ with returns $r_i(t)$. Assume that under the Capital Asset Pricing Model (CAPM) these returns are a function of the risk-free rate r_f, beta β_i, mean-zero and constant variance error $\epsilon_i(t)$, and market return $r_M(t)$; *i.e.*,

$$r_i(t) - r_f = \beta_i\left(r_M(t) - r_f\right) + \epsilon_i(t)$$

Also, the variance of the market return is ${\sigma_M}^2$.

You are next given the portfolio optimization problem over these n assets

$$\mathcal{M} = \min\left\{\frac{1}{2}\,\mathbf{x}^{\mathrm{T}}\,\boldsymbol{\Sigma}\,\mathbf{x} - \lambda\,\boldsymbol{\mu}^{\mathrm{T}}\,\mathbf{x}\;\middle|\;\mathbb{1}^{\mathrm{T}}\,\mathbf{x} = 1\right\}$$

where $\boldsymbol{\mu}$ is the vector of mean returns, $\boldsymbol{\Sigma}$ is the covariance matrix of returns, $0 \le \lambda \le \infty$ is a risk-reward trade-off parameter, and $\mathbb{1}$ is a vector of ones.

- Express the vector of mean returns in terms of the CAPM parameters.
- Express the covariance matrix in terms of the CAPM parameters.
- Derive in terms of the parameters of the CAPM a closed-form solution to proportionality to the mathematical program $\mathcal{M}$.
- Under the assumption $\beta_i > 0,\ \forall\, i$ what short positions can arise in the solution to $\mathcal{M}$?

6.

You are given the returns of $N = 100$ assets over $T = 250$ time periods.

- Compute the parameter q for the Marchenko-Pastur distribution of eigenvalues for a correlation matrix of uncorrelated assets for an estimation problem of this type.
- Compute the lower and upper bound for the associated Marchenko-Pastur distribution given q.
- You are given the partial list of sorted eigenvalues of the sample correlation matrix: $\{15.2, 8.2, 4.2, 3.1, 2.8, 2.2, 1.8, 1.6, 1.5, 1.4, 1.3, \ldots\}$. Based solely on the distribution (without any adjustment for sample size), which eigenvalues appear to be statistically meaningful?

7. **Generalized Itô Integral and Functional Calculus**. (30% + 30% + 40%) Let $f : \mathbb{R} \times [0, T] \to \mathbb{R}$ be a function such that $f \in C^{2,1}(\mathbb{R} \times [0, T])$, and let B_t be a standard Brownian motion. Define the process: $X_t := f(B_t, t)$.

(a) Use Itô's formula to compute $\mathrm{d}X_t = \mathrm{d}f(B_t, t)$ and express it as:

$$\mathrm{d}f(B_t, t) = \left(\frac{\partial f}{\partial t}(B_t, t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(B_t, t) \right) \mathrm{d}t + \frac{\partial f}{\partial x}(B_t, t)\mathrm{d}B_t.$$

(b) Multiply both sides of the expression in (a) by a process $g_t = B_t$, and derive the identity:

$$\int_0^t f(B_s, s)\,\mathrm{d}B_s = B_t f(B_t, t) - \int_0^t \left[B_s \frac{\partial f}{\partial t}(B_s, s) + \frac{\partial f}{\partial x}(B_s, s) + \frac{1}{2} B_s \frac{\partial^2 f}{\partial x^2}(B_s, s) \right] \mathrm{d}s - \int_0^t B_s \frac{\partial f}{\partial x}(B_s, s)\,\mathrm{d}B_s.$$

(c) Suppose now $f(B_t, t)$ is replaced by $f(X_t, t)$, where $X_t = \mu t + \sigma B_t$ is Brownian motion with drift. Derive an expression for $\int_0^t f(X_s, s)\,\mathrm{d}s$, using the generalized Itô formula for $f(X_t, t)$, and identify the new drift term.

8. **Chooser option**. (60% + 40%) Consider a model with a unique risk-neutral measure $\widetilde{\mathbb{P}}$ and constant interest rate r. According to the risk-neutral pricing formula, for $0 \le t \le T$, the price at time t of a European call expiring at time T is $C(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T)-K)^+ \,\middle|\, \mathcal{F}(t)\right]$, where $S(T)$ is the underlying asset price at time T and K is the strike price of the call. Similarly, the price at time t of a European put expiring at time T is $P(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}(K-S(T))^+ \,\middle|\, \mathcal{F}(t)\right]$. Finally, because $e^{-rt}S(t)$ is a martingale under $\widetilde{\mathbb{P}}$, the price at time t of a forward contract for delivery of one share of stock at time T in exchange for a payment of K at time T is

$$F(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T)-K) \,\middle|\, \mathcal{F}(t)\right] = S(t) - e^{-r(T-t)}K.$$

Because $(S(T)-K)^+ - (K-S(T))^+ = S(T)-K$, we have the *put-call parity* relationship:

$$\begin{aligned} C(t) - P(t) &= \widetilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T)-K)^+ - e^{-r(T-t)}(K-S(T))^+ \,\middle|\, \mathcal{F}(t)\right] \\ &= \widetilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T)-K) \,\middle|\, \mathcal{F}(t)\right] = F(t). \end{aligned}$$

Now consider a date t_0 between 0 and T, and consider a *chooser option*, which gives the right at time t_0 to choose to own either the call or the put.

(i) Show that at time t_0 the value of the chooser option is

$$C(t_0) + \max\{0, -F(t_0)\} = C(t_0) + \left(e^{-r(T-t_0)}K - S(t_0)\right)^+.$$

(ii) Show that the value of the chooser option at time 0 is the sum of the value of a call expiring at time T with strike price K and the value of a put expiring at time t_0 with strike price $e^{-r(T-t_0)}K$.

9. **(American Options)**.

(i) (30%) Consider a game where a player simultaneously throws one fair coin and one fair six-sided die. After each throw:

- If the coin shows **heads**, the player receives twice the number shown on the die.
- If the coin shows **tails**, the player loses one dollar (i.e., receives $-\$1$).

The player is allowed to throw the coin and die up to three times in total. After each throw, the player may choose to stop the game, in which case the accumulated reward is locked in and no further throws occur. The total reward is the cumulative sum of outcomes up to the chosen stopping time.

Compute the expected value of the game at time $n = 0$ (before the first throw), assuming the player follows an optimal stopping strategy.

(ii) $(30\% + 20\% + 20\%)$ Consider a frictionless financial market consisting of a risk-free asset and a risky stock, whose prices at time t are denoted B_t and S_t, respectively. Assume the price processes evolve as follows:

$$dB_t = rB_t\,dt, \qquad dS_t = \mu S_t\,dt + \sigma S_t\,dW_t,$$

where $r > 0$ is the constant risk-free interest rate, $\mu \in \mathbb{R}$ is the drift of the stock, $\sigma > 0$ is the stock volatility, $\{W_t\}_{t\geq 0}$ is a standard Brownian motion under the physical measure $\mathbb{P}$, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions. Assume the market is arbitrage-free and complete, and that pricing is carried out under the risk-neutral measure $\widetilde{\mathbb{P}}$, where the drift of the stock is replaced by r. Consider the time-homogeneous value function $V(S)$ of a perpetual American option (i.e., an American option with no expiry date), which has not yet been exercised.

(a) Show that in the continuation region, the function $V(S)$ satisfies the second-order ordinary differential equation:

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + rS\frac{dV}{dS} - rV(S) = 0.$$

(b) Show that the general solution of this ODE is given by: $V(S) = AS^{\alpha_+} + BS^{\alpha_-}$, where A and B are constants, and $\alpha_+ > 1$, $\alpha_- < 0$. Derive explicit expressions for α_+ and α_- in terms of r and σ.

(c) Use appropriate boundary and smooth-pasting conditions to derive the value function of a perpetual American **put** option with strike $K > 0$, assuming the underlying stock pays no dividends.

10. **(Gaussian copula and correlated default)**. (20% + 20% + 30% + 30%) Suppose that the default indicators are modeled using a **latent variable model**, $D_i = 1_{\{Z_i \le \Phi^{-1}(p_i)\}}$, where $(Z_1, Z_2) \sim N(0, \Sigma)$, and Σ is a covariance matrix with correlation $\rho \in (-1, 1)$. This induces a Gaussian copula for (D_1, D_2).

   (a) Define the Gaussian copula $C_\rho(u, v)$, and show

   $$P(D_1 = 1, D_2 = 1) = C_\rho(p_1, p_2) = \Phi_2(\Phi^{-1}(p_1), \Phi^{-1}(p_2); \rho),$$

   where $\Phi_2(\cdot, \cdot; \rho)$ is the CDF of the bivariate standard normal with correlation ρ.

   (b) Let $p_1 = p_2 = p$. Use the Freéchet bounds for copulas to show that $\max(0, 2p - 1) \le P(D_1 = 1, D_2 = 1) \le p$.

   (c) Show that the default correlation is

   $$\text{Corr}(D_1, D_2) = \frac{C_\rho(p_1, p_2) - p_1 p_2}{\sqrt{p_1 p_2 (1 - p_1)(1 - p_2)}}.$$

   (d) Show that the tail dependence coefficient for the Gaussian copula is zero when $\rho < 1$. What implication does this have for **joint extreme defaults**?

11. **(EGARCH model and VaR)**. (30% + 30% + 40%) Consider the following EGARCH(1,1) model,

$$r_t = \mu + \epsilon_t, \quad \epsilon_t = \sigma_t z_t, \quad \log \sigma_t^2 = \omega + \beta \log \sigma_{t-1}^2 + \alpha(|z_t| - E|z_t|) + \gamma z_t,$$

in which z_t are independent and identically distributed $N(0,1)$ random variables.

(a) Compute the kurtosis of the r_t.

(b) Compute the 1-step ahead VaR of r_t at level $q = 1\%$.

(c) Compute the 2-step ahead VaR of r_t at level $q = 2\%$.

12. **(Marshall-Olkin copula)**. (30% + 30% + 40%) Let T_1 and T_2 be non-negative random variables with a Marshall-Olkin bivariate exponential distribution, defined via independent exponential components, $T_1 = \min(X_1, X_0)$, and $T_2 = \min(X_2, X_0)$, where $X_i \sim Exp(\lambda_i)$, $i = 0, 1, 2$ and X_i are independent.

    (a) Derive the joint survival function $P(T_1 > t_1, T_2 > t_2)$.

    (b) Show that the associated copula is $C^{MO}(u, v) = \min(u^{1-\theta}, v^{1-\theta})(uv)^{\theta}$ for $\theta = \lambda_0/(\lambda_0 + \lambda_1 + \lambda_2)$.

    (c) Use the definition of the Kendall's tau to show that $\tau = P(T_1, T_2) = \theta$.